Homotopy Shields over Sets in $\mathbb{R}^n,\ n \in \mathbb{Z}^+$

Philipp Harland

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Email: piranhafisherman@proton.me

Abstract

In this paper, we will be defining the concept of a "homotopy shield" over a set in $\mathbb{R}^n$, and then documenting some immediate properties of its definition w.r.t. both binary and multiary contexts.

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1 Introduction

Let S be any path connected set with nonzero Lebesgue measure in $\mathbb{R}^n$. For the rest of the paper, this is what we will mean by "set" in the rest of this paper.

Def. 1.1. *An algebraic structure, S, which is defined over a parameter space $\mathcal{P}$, is called a shield if the variety it belongs to varies w.r.t. different parameters. Or, stated succinctly, $|\bigcup_{p\in\mathcal{P}} Var(S_p)| \geq 2$.* [1]

We can consider our "homotopy shields" to be defined over the parameter space $[0, 1]$.

Def. 1.2. *A function, f which is defined as a function – given S and T are points, or, they are collections of points of the same "type", e.g. 2 Jordan curves – is called a homotopy function if it is continuous, and, satisfies the condition that $f(0) = S,\ f(1) = T$.*

This condition is slightly different to how topologists define it, but it fits for our purposes.

[1] Here, instead of the context in algebraic geometry, we use the context in universal algebra where "algebraic variety" means a class of algebraic structures which all satisfy the same axioms, e.g. the class of groups, $\mathfrak{C}_{Grp}$ or the class of abelian groups, $\mathfrak{C}_{AbGrp}$.

2 Results of Research

2.1 As a binary operation

We let S be a set over $\mathbb{R}^n$, for a fixed positive integer n. We will set up the algebraic structure (S, f_t), where f_t is a homotopy function, which has a fixed interpolation value, t. We interpret this function as one belonging to the function space $S \times S \times \mathbb{I} \to S$.
We now document a few properties of homotopies on sets:
Thm. 2.1.1. *When $t = \frac{1}{2}$ exactly, we obtain an abelian structure under the linear homotopy.*
Proof. For 2 points $x = (x_1, ..., x_n),\ y = (y_1, .., , y_n) \in \mathbb{R}^n$, invoking the fact addition in $\mathbb{R}^n$ is commutative, thus $x+y = y+x = (x_1+y_1, ..., x_n+y_n)$, we have:
$f_{\frac{1}{2}}(x, y) = \frac{(x_1+y_1,...,x_n+y_n)}{2}$
$f_{\frac{1}{2}}(y, x) = \frac{(y_1+x_1,...,y_n+x_n)}{2}$ And, we know these two are equal, given the identity above.
Thm. 2.1.2. *We usually cannot define such structures on Lebesgue measure zero sets, but there are special cases where we can, i.e. the closure property is satisfied. These are the cases when $t \in \{0, 1\}$.*
Proof. Given two points, $x, y \in \mathbb{R}$ we know $0y + x = x$, and $0x + y = y$. And, since $t \notin (0, 1)$, it is impossible to get an 'in between' point.
Thm. 2.1.3. *Given two sets, A and B, $A +_{Mink} B = 2 \cdot \bigcup_{(a,b)\in A\times B} f_{\frac{1}{2}}(a, b)$. Here, by $n \cdot A$ we mean the set $\{(na_1, ..., na_k)|(a_1, ..., a_n) \in A\}$.*
Proof. This is trivial to prove.
For left and right identities, we will first describe such elements for the special cases $t \in \{0, \frac{1}{2}, 1\}$, then move onto the general case.
Thm 2.1.4. *When $t = \frac{1}{2}$, we have an identity for every element, a, of our underlying set, A, which is a itself.*
Proof. This is trivial to prove.
For homotopy shields over convex vs. concave sets, we have the following 2 theorems:
Thm. 2.1.5. *Over convex sets, homotopy shields are always closed.*
Proof. This follows immediately from the definition of a convex set.
Thm. 2.1.6. *Over concave sets, homotopy shields are only closed if $t \in \{0, 1\}$.*
Proof. This is trivial to prove.

2.2 As a k-ary operation

We can easily extend this theory to k-ary functions. Instead of 2, for a positive integer, k, we give the following theory:
Def 2.2.1. *Drawing from* **Def. 1.2.**, *we define a k-ary homotopy function as a continuous function; and, given k points, or, sets of points $A_1, ..., A_k$ of the same "type", we have:*

$$\begin{cases} f(1,...,0) & A_1 \\ . \\ . \\ . \\ f(0,...,1) & A_k \end{cases}$$

, where $f(0,...,1,...,0)$ with the i^{th} input equal to 1 maps to A_i. We, for a function which we denote $f_{t_1,...,t_{k-1}}$ will have our "interpolation value" be fixed. We view it as belonging to the function space $S^k \times \mathbb{I}^{k-1} \to S$.
Similarly to how we define linear homotopy functions in one variable, we can explicitly define such a function: $f_{(t_1,...,t_{k-1})}(x_1,...,x_k) = (1 - \sum_{m=1}^{k-1} t_m)x_k + \sum_{i=1}^{k-1} t_i x_i$. Here, all t_i's are non-negative, and sum to 1. We are aware of the fact that t_k is completely determined by all t_i, $i \in \{1,...,k-1\}$.
Thm. 2.2.1. *Similar to* **Thm. 2.2.** *– we can always define a "homotopy shield" over any set, even if its Lebesgue measure is zero if our function's "interpolation value" is among one of the cases in the second condition of* **Def. 2.1.**. *Proof.* This is trivial to prove using the proof of **Thm. 2.2.**.
Thm. 2.2.2. *Similarly to* **Thm. 2.4.**, *when our interpolation tuple is equal to $(\frac{1}{k},...,\frac{1}{k})$, there exists a double-sided identity element, a, for every element in our underlying set, which is $\underbrace{f_{(\frac{1}{k},...,\frac{1}{k})}(a,...,a)}_{k}$.*
Proof. This is trivial to prove.
Thm. 2.2.3. *Similarly to* **Thm. 2.3.** *Given n sets, $G_1,...,G_n$, $\dot{+}_{j=1}^{n} G_j = n \cdot \bigcup_{a_i \in G_i} f_{(\frac{1}{n}...\frac{1}{n})}(a_1,..,a_n)$.*
Proof. This is trivial to prove.
We have the following theorems which come as analogues to **Thm. 2.1.5.** *and* **Thm. 2.1.6.**:
Thm. 2.2.4. *Over convex sets, homotopy shields in k variables are always closed.*
Proof. Similarly to the analogue, this is trivial to prove. It comes directly from the definition of a convex set.
Thm. 2.2.5. *Over concave sets, homotopy shields are only closed if their interpolation values are equivalent to the natural basis vectors in $\mathbb{R}^{k-1}$.*
Proof. Similarly to the analogue, this is trivial to prove.[2]

[2]Here, by trivial, we mean it is very easy and doesn't take that much ingenuity or too many additional concepts to prove. This applies to all theorems which have been marked as "trivial".